ETMAG

CORONALECTURE 10

Matrices – rank of

Systems of linear equations

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I have included an additional example (slides 14 and 15)

and an answer to a FAQ about multiple EROS in slide 16.

Definition.

Let A be an $m \times n$ matrix over $\mathbb{K}$. *Row rank* of A is the dimension of the subspace of $\mathbb{K}^n$ spanned by rows of A. *Column rank* of A is the dimension of the subspace of $\mathbb{K}^m$ spanned by the columns of A.

Theorem.

For every matrix A the row rank of A is equal to its column rank.

For every matrix A *rank of A* or *r(A)* denotes both, the row rank and the column rank of A

Theorem.

If A is row equivalent to B (*i.e. A can be row-reduced to B*) then

$$r(A) = r(B)$$

Proof. It is enough to prove EROS (Elementary Row OperationS) do not affect the *space* spanned by rows, so they also do not affect the *dimension of the space*. It is trivial for interchanging and scaling of rows, i.e. EROS 1 and 2.

Comprehension (1 bonus point)**.**

Prove that EROS ($r_i \leftarrow r_i + cr_j$) does not affect the dimension of the vector space spanned by rows. In other words, prove

$$\dim(\ \mathit{span}\ (r_1, r_2, \ldots, r_i, \ldots, r_m)\ ) = \dim(\ \mathit{span}\ (r_1, r_2, \ldots, r_i + cr_j, \ldots, r_m)\ )$$

Theorem.

The rank of a matrix A is equal to the number of nonzero rows in any row-echelon matrix B which is row-equivalent to A.

Proof.

It follows from the last theorem and the fact that rows of a row-echelon matrix are linearly independent.

Remark. The theorem can be used as a tool for checking linear independence of a set of vectors. Form a matrix with the vectors serving as rows and calculate its rank. If the rank matches the number of the vectors, the set is linearly independent.

SYSTEMS OF LINEAR EQUATIONS

A system of linear equations

$$(*)\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \ \dots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ \dots + a_{2,n}x_n = b_2 \\ \cdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ \dots + a_{m,n}x_n = b_m \end{cases}$$

can be represented as a single matrix equation AX=B, where A = [$a_{i,j}$], $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$. X and B are single-column matrices.

The system of linear equations $(*)$ can also be represented as a vector equation

$$x_1 \underset{C_1}{\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}} + x_2 \underset{C_2}{\begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix}} + \ldots + x_n \underset{C_n}{\begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}} = \underset{B}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

We are trying to express B as a linear combination of columns of the coefficient matrix A. This can only be done if

$$\operatorname{span}\{C_1, C_2, \ldots, C_n\} = \operatorname{span}\{C_1, C_2, \ldots, C_n, B\}.$$

The matrix with columns $C_1, C_2, \ldots, C_n$ and B is called *the augmented matrix* of the system of equations and is denoted by [A|B].

Theorem. (Kronecker, Capelli)

A system $AX = B$ of linear equations has a solution iff

$$r(A) = r([A|B]).$$

Proof. The vector-oriented approach form the previous slide is proof enough.

Remark.

Interchanging equations, multiplying both sides by a non-zero number and adding equations one to another do not affect the set of solutions of a system of equations. EROS are exactly these operations except that they are performed on rows of a matrix rather than on equations. This suggests a strategy for solving a system of equations. Start with a system $(*)$, represent it as its augmented matrix $[A|B]$, row-reduce the matrix to a row echelon matrix $[E|C]$, translate the matrix to the language of equations.

Now move to slide 11 of the other presentation to see an example.

Definition.
A row echelon matrix is called *row canonical* iff the leading nonzero entry in each nonzero row is equal to 1 and is the only nonzero entry in its column.

Fact. Every matrix can be row reduced to a row canonical one.

Fact. If we row reduce [A,B] to a row canonical matrix rather than to a row echelon one we are given the solutions on a plate.

Example. Consider the reduced system from last example

$$\begin{cases} 2x + 4y - z = 11 \\ \quad\quad 5y + z = 2 \\ \quad\quad\quad 3z = -9 \end{cases}$$

Its augmented matrix is $\begin{bmatrix} 2 & 4 & -1 & 11 \\ 0 & 5 & 1 & 2 \\ 0 & 0 & 3 & -9 \end{bmatrix}$.

Doing $\frac{1}{3}r_3$ and then r_1+r_3 and r_2-r_3 we get $\begin{bmatrix} 2 & 4 & 0 & 8 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix}$. Then doing $\frac{1}{5}r_2$ and $\frac{1}{2}r_1$ we get $\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$. Finally, by r_1-2r_2 we get

$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ which in the language of equations means

$$\begin{cases} x \quad\quad\quad = 2 \\ \quad y \quad\quad = 1 \\ \quad\quad z = -3 \end{cases}$$

Definition.

A system of linear equations AX=B is called *homogeneous* iff B=Θ.

Fact.

Every homogeneous system of linear equations has a solution, namely $x_1 = 0$, $x_2 = 0$, … , $x_n = 0$. Any other solution (if there is one) is called a *non-trivial* or *non-zero* solution.

Theorem.

Let $AX=\Theta$ be a homogeneous system of *m* linear equations with *n* unknowns. Then the set $W = \{v \in \mathbb{K}^n \mid Av = \Theta\}$ of all solutions of the system is a subspace of the vector space $\mathbb{K}^n$. Moreover,

$\dim(W) = n - r(A)$.

Proof. (of the first statement)

Take $u, v \in W$. This means that $Au = \Theta$ and $Av = \Theta$. Since matrix multiplication is distributive over addition we have $A(u+v) = Au+Av = \Theta + \Theta = \Theta$, i.e. $u+v \in W$.

Similarly, we prove that for every $k \in \mathbb{K}$ we have $A(ku) = k(Au) = k\Theta = \Theta$.

We skip the proof of the second statement.

Example.

$$\begin{cases} x + y - z = 0 \\ 2x - 3y + z = 0 \\ x - 4y + 2z = 0 \end{cases} \quad A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 1 & -4 & 2 \end{bmatrix} \sim r_2\text{-}2r_1,\ r_3\text{-}r_1 \sim$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{bmatrix} \sim r_3\text{-}r_2 \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$ The rank of the last matrix is, clearly, 2. Hence the dimension of the solution space is 3 – 2 = 1.

We shall find a basis for the space reducing the matrix further.

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \frac{1}{-5} r_2 \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & \frac{-3}{5} \\ 0 & 0 & 0 \end{bmatrix} \sim r_1\text{-}r_2 \sim \begin{bmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{-3}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

In the language of equations this reads

$$\begin{cases} x + \frac{2}{5}z = 0 \\ y - \frac{3}{5}z = 0 \\ 0z = 0 \end{cases}$$

. The bottom equation really says "z may be anything you like" and the top two say $x = -\frac{2}{5}z$ and $y = \frac{3}{5}z$. Hence every vector (x,y,z) belonging to the solution space looks like $(-\frac{2}{5}z, \frac{3}{5}z, z) = z(-\frac{2}{5}, \frac{3}{5}, 1)$ and the set $\{(-\frac{2}{5}, \frac{3}{5}, 1)\}$ is a one-element basis for the space.

Example.

$$\begin{cases} x + ay + az = 1 \\ ax + ay + z = 1 \\ ax + y + az = 1 \\ ax + ay + az = 1 \end{cases}$$ Discuss solvability of the system in terms of a.

$$\begin{bmatrix} 1 & a & a & 1 \\ a & a & 1 & 1 \\ a & 1 & a & 1 \\ a & a & a & 1 \end{bmatrix} \sim \begin{matrix} r_2 - ar_1 \\ r_3 - ar_1 \\ r_4 - ar_1 \end{matrix} \sim \begin{bmatrix} 1 & a & a & 1 \\ 0 & a - a^2 & 1 - a^2 & 1 - a \\ 0 & 1 - a^2 & a - a^2 & 1 - a \\ 0 & a - a^2 & a - a^2 & 1 - a \end{bmatrix}$$. Since for a=1 everything outside the top row becomes equal to zero it looks like a good idea co split cases.

Case 1, a=1. We get $\begin{bmatrix} 1 & a & a & 1 \\ 0 & a - a^2 & 1 - a^2 & 1 - a \\ 0 & 1 - a^2 & a - a^2 & 1 - a \\ 0 & a - a^2 & a - a^2 & 1 - a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In this case r(A) = a(A|B) = 1. This means the system is solvable. It is reduced to $x + y + z = 1$, hence $x = 1 - y - z$, y and z are free.

Case 2, a≠1.

$$\begin{bmatrix} 1 & a & a & 1 \\ 0 & a-a^2 & 1-a^2 & 1-a \\ 0 & 1-a^2 & a-a^2 & 1-a \\ 0 & a-a^2 & a-a^2 & 1-a \end{bmatrix} \sim \mathit{divide\ rows}\ 2,3, \mathit{and}\ 4\ \mathit{by} (1-a) \sim$$

$$\begin{bmatrix} 1 & a & a & 1 \\ 0 & a & 1+a & 1 \\ 0 & 1+a & a & 1 \\ 0 & a & a & 1 \end{bmatrix} \sim \textit{subtract row 4 from other rows} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & a & 1 \end{bmatrix} \sim r_2 \leftrightarrow r_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & a & a & 1 \end{bmatrix} \sim r_4 - ar_2 - ar_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

r(A) = 3, r(A|B) = 4. The system is inconsistent.

FAQ.

Can we do several EROS in one step?

It depends. A common mistake is to do something like r_1-r_2 and r_3-r_1 in one go. What is wrong with this? Row r_1 has been modified in the first operation which means in the second one you should use the modified r_1. In extremal cases people are able to row-reduce any matrix to all zeros, like this:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \mathrm{r}_1\text{-}\mathrm{r}_2\,,\ \mathrm{r}_2\text{-}\mathrm{r}_1 \sim \begin{bmatrix} a-c & b-d \\ c-a & d-b \end{bmatrix} \sim \mathrm{r}_1+\mathrm{r}_2\,,\ \mathrm{r}_2+\mathrm{r}_1 \sim \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In short, when in doubt do it one EROS at a time.

Theorem.

Let AX = B be an arbitrary system of linear equations. Let U be the solution set and let $v_0 \in$ U be any single solution to the system.

Then U = v_0 +W = {v_0 + w | $w \in$ W}, where W is the solution space of the corresponding homogeneous system AX = Θ.

Proof.

Each vector w from v_0+W is a solution to AX = B. Indeed,

A(v_0 + w) = Av_0 + Aw = B + Θ = B.

Moreover, if, for some v, Av = B we can denote $t = v - v_0$.

We see that $v = v_0 + t$ and At = A($v - v_0$) = Av −Av_0 = B−B = Θ.

Illustration.

(1) $\{-x+y = 1$ (a system of equation, one equation two unknowns)

(2) $\{-x+y = 0$ (the corresponding homogeneous system)

v_0 some solution of (1)